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Difference equation in the space of holomorphic functions of exponential type and Ramanujan summation

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1 Introduction

In this report we will consider the following difference equation:

$$R(x+1) - \beta R(x) = A(x).$$

In [2] R.C.Buck studied this difference equation in the space of entire functions of exponential type. Using Avannisian-Gay transform and Fourier-Borel transform of analytic functionals with non-compact carriers, we will study difference equations in the space of holomorphic functions of exponential type defined in the right half plane. Our result is a generalization of C.R.Buck's result. For $\beta = 1$, our work is closely related to Ramanujan summation studied by Candelpergher,Coppo and Delabaere ([3]). In §5 we will explain the relation between our results and their results. In final section we will apply our results to Ramanujan summation.

2 Candelpergher-Coppo-Delabaere's method

According to [3], we introduce Candelpergher-Coppo-Delabaere's method to solve the following difference equation :

$$R(x+1) - R(x) = A(x), \qquad (D)$$

2-1. Formal series expansion of solution to (D)

E denotes the following infinite differential operator(translation operator):

$$E = e^{\partial_x} = \sum_{n=0}^{\infty} \frac{\partial_x^n}{n!}.$$

$$Ef(x) = \sum_{n=0}^{\infty} \frac{\partial_x^n}{n!} f(x) = f(x+1).$$

Using operator E, difference equation

$$R(x+1) - R(x) = A(x)$$

becomes to

$$-(I - e^{\partial_x})R(x) = A(x),$$

where I denotes identity operator. Hence we have

$$R(x) = \frac{1}{e^{\partial_x} - I} A(x) = \frac{\partial_x}{e^{\partial_x} - I} {\partial_x}^{-1} A(x).$$

Now we use following Taylor expansion

$$\frac{z}{e^z - 1} = 1 + \sum_{k=1}^{\infty} \frac{B_k}{k!} z^k, \qquad (|z| < 2\pi),$$

where B_k denotes Bernoulli number. For example,

$$B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}.$$

For the details of Bernoulli numbers, we refer the reader to [4]. Making use of this Taylor expansion, we have

$$\begin{aligned} R(x) &= \frac{\partial_x}{e^{\partial_x} - I} {\partial_x}^{-1} A(x) \\ &= (I + \sum_{k=1}^{\infty} \frac{B_k}{k!} {\partial_x}^k) {\partial_x}^{-1} A(x) \\ &= \int A(x) dx + \sum_{k=1}^{\infty} \frac{B_k}{k!} {\partial_x}^{k-1} A(x). \end{aligned}$$

This is formal series expansion of the solution to difference equation (D).

Example 1.(Stirling's formula [4]) Put $A(z) = \log z$. $R(z) = \log \Gamma(z)$ satisfies the difference equation $R(z+1) - R(z) = \log z$. We have

$$\log \Gamma(z) = z \log z - z + C - \frac{1}{2} \log z + \sum_{k=2}^{\infty} \frac{B_k}{k(k-1)} (-1)^{k-2} z^{-k+1},$$

where C is a constant $(= \frac{1}{2}\log 2\pi)$.

2-2.Integral representation of the solution to difference equation (D).

Suppose that $A(x)$ has Laplace integral representation :

$$A(x) = \int_\gamma e^{-xz}\hat{A}(z)dz,$$

where γ is suitable countour. Then we have

$$\begin{aligned} R(x) &= \int A(x)dx + \sum_{k=1}^{\infty} \frac{B_k}{k!}\partial_x{}^{k-1}A(x) \\ &= \int A(x)dx + (\sum_{k=1}^{\infty} \frac{B_k}{k!}\partial_x{}^{k-1}) \int_\gamma e^{-xz}\hat{A}(z)dz \\ &= \int A(x)dx + \sum_{k=1}^{\infty} \frac{B_k}{k!} \int_\gamma (-z)^{k-1}e^{-xz}\hat{A}(z)dz \\ &= \int A(x)dx - \int_\gamma \left(\frac{1}{1-e^{-z}} - \frac{1}{z}\right) e^{-xz}\hat{A}(z)dz \end{aligned}$$

This is the integral representation of the solution of the difference equation(D).

Example 2(Binet's formula [4]) We put $A(z) = \frac{1}{z}$. $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ satisfies difference equation $R(z+1) - R(z) = \frac{1}{z}$. In this example $\hat{A}(z) = 1$ and $\gamma = [0, \infty)$. So we have

$$\psi(z) = \log z - \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right) e^{-zt}dt, \qquad (Re(z) > 0).$$

The relation between the solution of difference equation (D) and Ramanujan summation will be explained in §5.

3 Transformations of analytic functionals with non-compact carriers

Let $L = [a, \infty) + \sqrt{-1}[-b, b]$. L_ε denotes the ε neighbourhood of L. We introduce following test function space.

$$Q(L;\tau) = \lim ind_{\varepsilon>0,\varepsilon'>0} Q_b(L_\varepsilon : \tau + \varepsilon'),$$

where $Q_b(L_\varepsilon : \tau+\varepsilon')$ denotes the space of functions which are holomorphic in the interior of L_ε , continuous in the closure of L_ε and satisfy following estimate:

$$sup_{t\in L_\varepsilon}|f(t)e^{(\tau+\varepsilon')t}| < +\infty.$$

$Q'(L:\tau)$ denotes the dual space of $Q(L:\tau)$. The element of $Q'(L:\tau)$ is called analytic functional carried by L and of type τ. For $T \in Q'(L:\tau)$ we define Fourier-Borel transform $\widetilde{T}(z)$ as follows;

$$\widetilde{T}(z) = \langle T_t, e^{-zt}\rangle, \qquad (Re(z) > \tau).$$

$Exp((\tau,\infty)+\sqrt{-1}R; L)$ denotes the space of holomorphic functions $F(z)$ defined in the right half plane $Re(z) > \tau$ satisfying following estimate: $\forall\varepsilon > 0, \forall\varepsilon' > 0, \exists C_{\varepsilon,\varepsilon'} \geq 0$,

$$|F(z)| \leq C_{\varepsilon,\varepsilon'}e^{H_L(z)+\varepsilon|z|}, \qquad (Re(z) \geq \tau+\varepsilon').$$

Following theorem characterizes Fourier-Borel transform of $Q'(L;\tau)$.

Theorem 1([5]) Fourier-Borel transform is a linear topological isomorphim from $Q'(L:\tau)$ to $Exp((\tau,\infty)+\sqrt{-1}R : L)$.

(Remark) The theory of analytic functionals with non-compact carrier is closely related to that of hyperfunction with exponential growth([8]).

If $\tau < 1$ and $0 \leq b < \pi$, then we can define Avanissian-Gay transform $G_T(w)$ as follows:

$$G_T(w) = \langle T_t, \frac{1}{1-we^t}\rangle.$$

Avanissian-Gay transform $G_T(w)$ has following properties.

Proposition 2([1],[6])

(1) $G_T(w)$ is holomorphic in $C \setminus exp(-L)$.

(2)

$$G_T(w) = -\sum_{n=1}^{\infty}\widetilde{T}(n)w^{-n}, \qquad (|w| > e^{-a}).$$

(3) $\forall\varepsilon > 0, \forall\varepsilon' > 0, \exists C_{\varepsilon,\varepsilon'} \geq 0$

$$|G_T(w)| \leq C_{\varepsilon,\varepsilon'}|w|^{-\tau-\varepsilon'}, \qquad (b+\varepsilon \leq |arg(w)| \leq \pi).$$

(4) (Integral representation of $\tilde{T}(z)$)

$$\tilde{T}(z) = \frac{-1}{2\pi\sqrt{-1}} \int_{\partial L_\epsilon} G_T(e^{-t})e^{-zt}dt.$$

$H_0(C\backslash exp(-L) : \tau)$ denotes the space of holomorphic functions which satisfy (1) (2),(3) in Proposition 2.

Following Carlson's theorem is an immediate consequence of Theorem 1 and Proposition 2.

Theorem 3(Carlson [6]) Suppose that $\tau < 1$ and $F(z) \in Exp((\tau, \infty) + \sqrt{-1}R : L)$ satisfies following condition:

$$F(n) = 0, \qquad (n = 1, 2...).$$

If $0 \leq b < \pi$, then $F(z)$ vanishes identically.

To end this section we give the following proposition which characterizes the sequences $\{F(n)\}_{n=1}^{\infty}$.

Proposition 4(Leroy-Lindelöf [7]) Suppose that $\tau < 1, 0 \leq b < \pi$. For sequence $\{A(n)\}_{n=1}^{\infty}$ following statements are equivalent.

(1) There exists $F(z) \in Exp((\tau, \infty) + \sqrt{-1}R : L)$ such that $A(n) = F(n), (n = 1, 2, ...)$.

(2)$\sum_{n=1}^{\infty} A(n)w^{-n}$ is analytically continued to $C \setminus exp(-L)$ and satisfies the conditions (3) in Proposition 2.

(Remark) We call $F(z)$ in prop.4 interpolating function for the sequence $\{A(n)\}_{n=1}^{\infty}$. By virtue of Carlson's theorem, there exists at most one interpolationg function for the sequence $\{A(n)\}_{n=1}^{\infty}$.

4 Main theorem

In this section we will prove our main theorem.

Main Theorem Suppose that $A(z) \in Exp((\tau, \infty) + \sqrt{-1}R : L)$, with $\tau < 1$ and $0 \leq b < \pi$. We consider the following difference equation

$$F(n+1) - \beta F(n) = A(n), \qquad (n = 1, 2, ...). \qquad (D_\beta).$$

(1) If β is not in negative real axis, then difference equation (D_β) has a solution in $Exp((\tau,\infty)+\sqrt{-1}R:L)$.
If $\beta=1$, then the solution is unique up to constant.
(2) If β is in negative real axis, then (D_β) has solution in $Exp((\tau,\infty)+\sqrt{-1}R:L)$ if and only if $G_S(\beta)=0$.
(3) If $A(z)$ is entire function of exponential type, then $F(z)$ is also entire function of exponential type.
(4)(Integral representation of solution of (D_β))

$$F(z)=\frac{-1}{2\pi\sqrt{-1}}\int_{\partial L_\epsilon}\frac{e^{-zt}G_S(e^{-t})}{e^{-t}-\beta}dt+C\beta^z,$$

where $A(z)=\tilde{S}(z)$,(Fourier-Borel transform of $S\in Q'(L:\tau)$), $G_S(w)$ is Avanissian-Gay transform of S and C is a constant.
(Proof of main theorem)
By theorem 1, there exists an analytic functional with non-compact carrier $S\in Q'(L:\tau)$ such that $\tilde{S}(z)=A(z)$. Suppose that

$$F(n+1)-\beta F(n)=A(n),\qquad (n=1,2..).$$

Multiply w^{-n} to both sides of this difference equation, we have

$$(w-\beta)\sum_{n=1}^{\infty}F(n)w^{-n}=\sum_{n=1}^{\infty}A(n)w^{-n}+C,$$

where C is some constant. So we have

$$-\sum_{n=1}^{\infty}F(n)w^{-n}=\frac{G_S(w)}{w-\beta}-\frac{C}{w-\beta}.$$

We put

$$F(z)=\frac{-1}{2\pi i}\int_{\partial L_\epsilon}\frac{G_S(e^{-t})}{e^{-t}-\beta}e^{-zt}dt.$$

Then $F(z)$ satisfies difference equation (D_β). Since

$$\frac{G_S(w)}{w-\beta}\in H_0(C\setminus exp(-L):\tau),$$

$F(z)\in Exp((\tau,\infty)+iR;L)$.
If $\beta=1$, then the solution of (D) is unique up to constant. This follows from Carlson's theorem.

5 The relation between Candelpergher-Coppo-Delabaere's results and our results.

In this section we will derive Candelpergher-Coppo-Delabaere's results from our results. We consider the difference equation (D).
First we will show that integral representation (4) in our main theorem coincides with integral representation obtained in section 2.
In previous section we obtained integral representation of the solution $F(z)$ of difference equation $R(z+1)-R(z)=A(z)$. Namely we have

$$F(z)=\frac{-1}{2\pi\sqrt{-1}}\int_{\partial L_\varepsilon}\frac{e^{-zt}G_S(e^{-t})}{e^{-t}-1}dt+C,$$

where $\widetilde{S}(z)=A(z)$ and C is a constant.
On the other hand Candelpergher-Coppo-Delabaere obtained following integral representation

$$R(z)=\int A(z)dz+\int_\gamma\left(\frac{1}{e^{-t}-1}+\frac{1}{t}\right)e^{-zt}\hat{A}(t)dt.$$

We put

$$f(z)=\int A(z)dz+\int_\gamma\frac{e^{-zt}}{t}\hat{A}(t)dt.$$

Then we have

$$\frac{df(z)}{dz}=A(z)+\int_\gamma(-t)\frac{e^{-zt}}{t}\hat{A}(t)dt=A(z)-A(z)=0$$

Hence $f(z)$ is constant. So

$$R(z)=\int_\gamma\frac{e^{-zt}}{e^{-t}-1}\hat{A}(t)dt+C.$$

This is equals to our integral representaion with $\gamma=L_\varepsilon$ and $\hat{A}(t)=-\dfrac{G_S(e^{-t})}{2\pi\sqrt{-1}}$.

Example 3(Complex integral form of Binet's formula) Put $A(z)=\dfrac{1}{z}$. $\psi(z)=\dfrac{\Gamma'(z)}{\Gamma(z)}$ satisfies the difference equation $R(z+1)-R(z)=\dfrac{1}{z}$. $A(z)=$

$\frac{1}{z}$ is Fourier-Borel transform of Heaviside function $H(t) \in Q'([0,\infty) : \{0\})$. $G_H(e^{-t}) = \log(1-e^t)$. So we have

$$\psi(z) = \log z + \frac{-1}{2\pi\sqrt{-1}} \int_\infty^{(0+)} e^{-zt} \left(\frac{1}{e^{-t}-1} - 1 \right) \log(1-e^t) dt.$$

This integral representation coincides with Binet's formula in example 2. Next we consider the series expansion of the solution of difference equation (D). We have

$$\frac{1}{e^{-t}-1} = -\frac{1}{t} + \sum_{k=1}^{\infty} \frac{B_k}{k!}(-t)^{k-1}.$$

On L_ε, we have

$$\left| \frac{1}{e^{-t}-1} - \left\{ -\frac{1}{t} + \sum_{k=1}^{\infty} \frac{B_k}{k!}(-t)^{k-1} \right\} \right| \leq C_n |t|^{n+1}.$$

where C_n is a constant depending on n.

$$\begin{aligned} F(z) &= \frac{-1}{2\pi\sqrt{-1}} \int_{\partial L_\varepsilon} e^{-zt} G_S(e^{-t}) \left(\frac{1}{e^{-t}-1} - \left\{ -\frac{1}{t} + \sum_{k=1}^{\infty} \frac{B_k}{k!}(-t)^{k-1} \right\} \right) dt \\ &+ \frac{-1}{2\pi\sqrt{-1}} \int_{\partial L_\varepsilon} e^{-zt} G_S(e^{-t}) \left\{ -\frac{1}{t} + \sum_{k=1}^{\infty} \frac{B_k}{k!}(-t)^{k-1} \right\} dt + C. \end{aligned}$$

The first integral is estimated by $C'_n |z|^{-n-1}$ and second integral is equals to

$$\sum_{k=1}^{n} \frac{B_k}{k!} \int_{\partial L_\varepsilon} e^{-zt} G_S(e^{-t})(-t)^{k-1} dt = \sum_{k=1}^{n} \frac{B_k}{k!} \frac{\partial^{k-1}}{\partial z^{k-1}} A(z).$$

So we have

$$\left| F(z) - \int A(z)dz - \sum_{k=1}^{n} \frac{B_k}{k!} \frac{\partial^{k-1}}{\partial z^{k-1}} A(z) \right| \leq C_n |z|^{-n-1}.$$

This gives the same formal expansion of solution to (D) obtained in §2.

6 Ramanujan transform and Ramanujan summation

According to [3] we explain Ramanujan transform and Ramanujan summation briefly. For the sequence $\{a(n)\}_{n=1}^{\infty}$, we put

$$R(n) = \sum_{k=n}^{\infty} a(k).$$

Then $R(n)$ satisfies

$$R(n) - R(n+1) = a(n), \qquad (n = 1, 2, ...).$$

And

$$R(1) = \sum_{k=1}^{\infty} a(k).$$

This is the basic idea to caluculate the infinite sum

$$\sum_{k=1}^{\infty} a(k).$$

Ramanujan sum $\sum_{n\geq 1}^{R} a(n)$ is defined as follows:
(i) Solve the difference equation

$$R(n) - R(n+1) = a(n), \qquad (n = 1, 2...).$$

with $\int_1^2 R(t)dt = 0$.
The solution of above difference equation is denoted by $R_a(t)$.
(ii) Calculate $R_a(1)$.
$R_a(1)$ is called Ramanujan sum of $\{a(n)\}_{n=1}^{\infty}$ and we put $R_a(1) = \sum_{n\geq 1}^{R} a(n)$.

$$R : a \to R_a$$

is called Ramanujan transform.
We put $L_0 = [0, \infty) + \sqrt{-1}[-b, b]$.
Proposition 5([3]) Suppose that b is less than π and there exists $\tilde{a}(z) \in Exp((\tau, \infty) + \sqrt{-1}R : L_0)$ such that $\tilde{a}(n) = a(n), (n = 1, 2, ...)$. Then following statements are valid.
(1)If $0 \in L$, then Ramanujan transform is a linear isomorphism of $Exp((\tau, \infty) + \sqrt{-1}R : L_0)$.
If $0 \notin L$, then Ramanujan transform is linear map from $Exp((\tau . \infty) + \sqrt{-1}R : L)$ to $Exp((\tau, \infty) + \sqrt{-1}R : L_o)$.
(2)If $a(n) = b(n), (n = 1, 2...)$ then

$$\sum_{n\geq 1}^{R} a(n) = \sum_{n\geq 1}^{R} b(n).$$

(3)If $\int_1^{\infty} |a(t)|dt < \infty$, then we have

$$\sum_{n\geq 1}^{R} a(n) = \sum_{n=1}^{\infty} a(n) - \int_1^{\infty} \tilde{a}(t)dt.$$

(Proof)
(1) is an immediate consequence of main theorem. By virtue of Carlson's theorem, interpolationg function for $\{a(n)\}$ and $\{b(n)\}$ are same. Hence we can conclude (2).
To prove (3) we make use of integral representation of solution to (D):

$$R(z) = \frac{-1}{2\pi\sqrt{-1}} \int_{\partial L_\epsilon} \frac{e^{-zt} G_S(e^{-t})}{e^{-t}-1} dt + C,$$

where $\widetilde{S}(z) = -\widetilde{a}(z)$ and C is a constant determined by the condition $\int_1^2 R(t)dt = 0$.

$$\begin{aligned} R(1) &= \frac{-1}{2\pi\sqrt{-1}} \int_{\partial L_\epsilon} \frac{e^{-t} G_S(e^{-t})}{e^{-t}-1} dt + C \\ &= G(1) + C \\ &= \sum_{n=1}^{\infty} a(n) + C. \end{aligned}$$

Now we calculate constant C which satisfies the condition $\int_1^2 R(t)dt = 0$.

$$\begin{aligned} 0 &= \int_1^2 R(t)dt = \frac{-1}{2\pi\sqrt{-1}} \int_{\partial L_\epsilon} \frac{G_S(e^{-t})}{e^{-t}-1} \int_1^2 e^{-zt} dz dt + C \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\partial L_\epsilon} G_S(e^{-t}) \frac{e^{-t}}{t} dt + C \end{aligned}$$

So

$$\begin{aligned} C &= \frac{-1}{2\pi\sqrt{-1}} \int_{\partial L_\epsilon} G_S(e^{-t}) \frac{e^{-t}}{t} dt \\ &= \frac{-1}{2\pi\sqrt{-1}} \int_{\partial L_\epsilon} G_S(e^{-t}) e^{-t} \int_0^{\infty} e^{-zt} dz dt \\ &= \int_0^{\infty} e^{-zt} \frac{-1}{2\pi\sqrt{-1}} \int_{\partial L_\epsilon} G_S(e^{-t}) e^{-t(1+z)} dz dt \\ &= -\int_0^{\infty} \widetilde{a}(1+z) dz = -\int_1^{\infty} \widetilde{a}(z) dz. \end{aligned}$$

Hence we have

$$R_a(1) = \sum_{n=1}^{\infty} a(n) - \int_1^{\infty} \widetilde{a}(z) dz.$$

Example 4. $a(n)=1$. In this example, $R_a(z)=\frac{3}{2}-z$.

$$R_a(1)=\sum_{n\geq 1}^{R} 1=\frac{1}{2}.$$

Remark that Ramanujan summation $\sum_{n\geq 1}^{R} 1=\frac{1}{2}$ is not equals to $\zeta(0)=\sum_{n=1}^{\infty} 1=-\frac{1}{2}$. ($\zeta(z)$ denotes Riemman zeta function).

Example 5. $a(n)=\dfrac{1}{n}$. In this example $R_a(z)$ is given by $-\psi(z)=-\dfrac{\Gamma'(z)}{\Gamma(z)}$,($\Gamma(z)$ denotes Euler Gamma function). It is well known that $\psi(1)=-\gamma$. Hence we have

$$R_a(1)=\sum_{n\geq 1}^{R} a(n)=\gamma.$$

(γ is Euler's constant).

Example 6. $a(n)=\dfrac{1}{n^2}$. In this example solution $R_a(z)$ is given by $\psi'(z)-1$. So we have

$$R_a(1)=\psi'(1)-1=\frac{\pi^2}{6}-1=\sum_{k=1}^{\infty}\frac{1}{k^2}-\int_1^{\infty}\frac{1}{t^2}dt.$$

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